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CONGRUENCE OF HYPERSURFACES OF A PSEUDO-EUCLIDEAN SPACE

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R. S. Kulkarni has proved [1] that the so-called bending of a hypersurface in an Euclidean space determines the congruence class of the hypersurface. In the present paper we show that a similar result holds for hypersurfaces of a pseudo-Euclidean space \mathbb{R}_s^{n+1} , $n > 2$. We prove also a corresponding theorem, which accounts for the behaviour of the second fundamental form on isotropic vectors.

1. PRELIMINARIES

Let M be a Riemannian or a pseudo-Riemannian manifold with a metric tensor g . A tangent vector ξ is said to be isotropic, if it is nonzero and $g(\xi, \xi) = 0$. Of course, for isotropic vectors one speaks only when the manifold is pseudo-Riemannian, i.e. when g is an indefinite metric. The values of a symmetric tensor of type (0,2) on isotropic vectors give a good information about this tensor, as it is shown by the following

Lemma 1 [2]. Let M be a pseudo-Riemannian manifold. If L is a symmetric tensor of type (0,2) on a tangent space $T_p M$, such that $L(\xi, \xi) = 0$ for every isotropic vector ξ in $T_p M$, then $L = cg$, where c is a real number.

Let ∇ and R denote the covariant differentiation and the curvature tensor of M , respectively. The Ricci tensor and the scalar curvature will be denoted by S and τ , respectively. Then the Weil conformal curvature tensor C for M is defined by

$$C = R - \frac{1}{n-2}\varphi + \frac{\tau}{(n-1)(n-2)}\pi_1 ,$$

1

where $n = \dim M$, φ is defined by

$$\varphi(T)(x, y, z, u) = g(x, u)T(y, z) - g(x, z)T(y, u) + g(y, z)T(x, u) - g(y, u)T(x, z) ,$$

for any symmetric tensor T of type (0,2) and $\pi_1 = \frac{1}{2}\varphi(g)$. As it is well known [3], if $n > 3$, then M is conformally flat if and only if the Weil conformal curvature tensor vanishes identically. If $n = 3$ a necessary and sufficient condition for M to be conformally flat is [3]

$$(1.1) \quad \left(\nabla_X \left(S - \frac{\tau}{4} g \right) \right) (Y, Z) - \left(\nabla_Y \left(S - \frac{\tau}{4} g \right) \right) (X, Z) = 0 .$$

If \overline{M} is another Riemannian or pseudo-Riemannian manifold, we denote the corresponding objects for \overline{M} by a bar overhead. Assume that f is a conformal diffeomorphism of M onto \overline{M} : $f^*\bar{g} = \varepsilon e^{2\sigma}g$, where $\varepsilon = \pm 1$ and σ is a smooth function. Then we have [3]

$$(1.2) \quad f^*\overline{R} = \varepsilon e^{2\sigma} \{ R + \varphi(Q) \} ,$$

where

$$(1.3) \quad Q(X, Y) = X\sigma Y\sigma - g(\nabla_X \nabla \sigma, Y) - \frac{1}{2} \|\nabla \sigma\|^2 g(X, Y) ,$$

$\nabla \sigma$ denoting the gradient of σ and $\|\nabla \sigma\|^2 = g(\nabla \sigma, \nabla \sigma)$.

In [4] we have proved the following

Lemma 2. Let M and \overline{M} be pseudo-Riemannian manifolds of dimension > 2 and f be a diffeomorphism of M onto \overline{M} . Assume that at a point p of M there exists an isotropic vector ξ , such that every isotropic vector, which is sufficiently close to ξ , is mapped by f_* in an isotropic vector in $f(p)$. Then f_* is a homothety at p .

In what follows M will be a hypersurface of an Euclidean space \mathbb{R}^{n+1} or of a pseudo-Euclidean space \mathbb{R}_s^{n+1} , such that the restriction g of the usual metric of \mathbb{R}_s^{n+1} to M is nondegenerate. Denote the second fundamental form of M by h . Then we have the equation of Gauss

$$R(X, Y, Z, U) = h(X, U)h(Y, Z) - h(X, Z)h(Y, U)$$

and the equation of Codazzi

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = 0 .$$

Recall also, that a point p of M is said to be quasi-umbilic, if

$$h = \alpha g + \beta \omega \otimes \omega$$

in p , where α, β are real functions and ω is a 1-form. In particular, if β is zero the point p is called umbilic.

The bending [1] K_h of M is said to be the function, assigning to each nonisotropic nonzero tangent vector x at a point of M the number

$$K_h(x) = \frac{h(x, x)}{g(x, x)} .$$

Two hypersurfaces M and \overline{M} being defined a diffeomorphism f of M onto \overline{M} is said to be bending preserving [1], if

$$(1.4) \quad \overline{K}_{\bar{h}}(f_*x) = K_h(x)$$

for each nonisotropic nonzero vector x on M , whose image is also nonisotropic. The analogue of (1.4) for isotropic vectors is

$$(1.5) \quad \lim_{x \rightarrow \xi} \frac{\overline{K}_{\bar{h}}(f_*x)}{K_h(x)} = 1 ,$$

where the isotropic vector ξ is approximated by nonisotropic nonzero vectors, whose images are also nonisotropic. We shall prove:

Theorem 1. Let M and \overline{M} be hypersurfaces with indefinite metrics in \mathbb{R}_s^{n+1} , $n > 2$, and let f be a diffeomorphism of M onto \overline{M} , satisfying (1.5) for each isotropic vector ξ on M . If the nonquasi-umbilic points are dense in M and if M is not conformally flat, then f is a congruence.

We recall that f is said to be a congruence if it can be extended to a motion of \mathbb{R}_s^{n+1} .

Theorem 2. Let M and \overline{M} be hypersurfaces with indefinite metrics in \mathbb{R}_s^{n+1} , $n > 2$, and let f be a bending preserving diffeomorphism of M onto \overline{M} . If the nonumbilic points are dense in M and the curvature tensor of M does not vanish identically in a point p , then there exists a neighbourhood V of p such that $f|_V$ is a congruence of V onto $f(V)$.

Remark. The proof of the congruence theorem in [1] can be applied for hypersurfaces with definite metrics (i.e. spacelike hypersurfaces) in \mathbb{R}_1^{n+1} .

2. BASIC RESULTS

In this section we prove two lemmas, which will be useful in the proofs of Theorems 1 and 2.

Lemma 3. Let M and \overline{M} be hypersurfaces with indefinite metrics in \mathbb{R}_s^{n+1} , $n > 2$, and let f be a diffeomorphism of M onto \overline{M} , satisfying (1.5) for each isotropic vector ξ on M . If the nonumbilic points are dense in M , then

- a) f is conformal: $f^*\bar{g} = \varepsilon e^{2\sigma}g$;
- b) $f^*\bar{h} = \varepsilon e^{2\sigma}\{h + \lambda g\}$, where λ is a smooth function;
- c) $f^*\bar{R} = e^{4\sigma}\{R + \lambda\varphi(h) + \lambda^2\pi_1\}$;
- d) the following equations hold:

$$(2.1) \quad X\sigma B(Y, Z) - Y\sigma B(X, Z) + \frac{1}{n}\{X\lambda g(Y, Z) - Y\lambda g(X, Z)\} = 0 ,$$

$$(2.2) \quad B(Y, \nabla\sigma) = \frac{n-1}{n}Y\lambda ,$$

where $B = h - \frac{\text{tr}h}{n}g$.

Proof. Let p be a nonumbilic point of M , i.e. h is not proportional to g in p . Then by Lemma 1 there exists an isotropic vector ξ in $T_p M$, such that $h(\xi, \xi) \neq 0$. Hence $h(\xi', \xi') \neq 0$ for each isotropic vector ξ , which is sufficiently close to ξ . Then (1.5) implies that $f_{*p}\xi'$ is isotropic. According to Lemma 2, f_* is a homothety at p . Since the nonumbilic points are dense in M , f is conformal and then a) is proved.

From a) and (1.5) it follows $(f^*\bar{h})(\xi, \xi) = \varepsilon e^{2\sigma} h(\xi, \xi)$. Applying again Lemma 1, we obtain b). Then c) follows from b) and from the equations of Gauss for M and \bar{M} .

To simplify the notations in the proof of d), we identify M with \bar{M} via f , and omit f^* from the formulas. Then we have [3]

$$\bar{\nabla}_X Y = \nabla_X Y + X\sigma Y + Y\sigma X - g(X, Y)\nabla\sigma .$$

Hence, using b) and the equations of Codazzi for g and \bar{g} , we find

$$(2.3) \quad \begin{aligned} X\sigma h(Y, Z) - Y\sigma h(X, Z) + X\lambda g(Y, Z) - Y\lambda g(X, Z) \\ + g(X, Z)h(Y, \nabla\sigma) - g(Y, Z)h(X, \nabla\sigma) = 0 , \end{aligned}$$

which implies immediately

$$(2.4) \quad h(Y, \nabla\sigma) = \frac{n-1}{n} Y\lambda + \frac{\text{tr } h}{n} Y\sigma ,$$

i.e. (2.2). From (2.3) and (2.4) we obtain (2.1). This proves the lemma.

We note that the conditions of Lemma 3 are fulfilled in Theorem 1, as well as in Theorem 2.

Lemma 4. If in Lemma 3 $\|\sigma\|^2 = 0$ and U denotes the open set $\{p \in M : \nabla\sigma \neq 0\}$, then

- a) each point of U is quasi-umbilic;
- b) $R = 0$ in U .

Proof. We shall use a connected component U_1 of U . Let in (2.1) $X = Z = \nabla\sigma$, $Y = \nabla\lambda$. By (2.2), we obtain

$$(2.5) \quad (\nabla\sigma)\lambda = 0 .$$

Now we put $X = \nabla\sigma$ in (2.1) and get use of (2.2) and (2.5). The result is $Y\sigma Z\lambda = 0$ for arbitrary vector fields Y, Z . Since $\nabla\sigma$ can not vanish in U_1 , it follows $\lambda = \text{const}$ (in U_1). Then (2.1) reduces to

$$X\sigma B(Y, Z) - Y\sigma B(X, Z) = 0 ,$$

which implies

$$(2.6) \quad B = \mu d\sigma \otimes d\sigma ,$$

where μ is a smooth function. Equivalently, we may write

$$(2.6') \quad h = \frac{\text{tr } h}{n} g + \mu d\sigma \otimes d\sigma ,$$

thus proving a). From the equation of Gauss for g it follows

$$S(x, x) = \text{tr } h \cdot h(x, y) - \sum_{i=1}^n h(x, e_i) h(y, e_i) g(e_i, e_i) ,$$

where $\{e_i; i, \dots, n\}$ is an orthogonal frame. Hence, using (2.6'), we obtain

$$(2.7) \quad S = \frac{n-2}{n} \mu \operatorname{tr} h d\sigma \otimes d\sigma + \frac{n-2}{n^2} (\operatorname{tr} h)^2 g .$$

Thus we get

$$(2.8) \quad \tau = \frac{n-1}{n} (\operatorname{tr} h)^2 .$$

From (2.7) and (2.8) we compute for $P = S - \frac{\tau}{n} g$:

$$(2.9) \quad P = \frac{n-2}{n} \mu \operatorname{tr} h d\sigma \otimes d\sigma .$$

By Lemma 3 c) we find immediately

$$(2.10) \quad \begin{aligned} f^* \bar{S} &= \varepsilon e^{2\sigma} \{ S + \lambda(n-2)h + \lambda \operatorname{tr} h \cdot g + (n-1)\lambda^2 g \} , \\ f^* \bar{\tau} &= \tau + 2(n-1)\lambda \operatorname{tr} h + n(n-1)\lambda^2 , \\ f^* \bar{P} &= \varepsilon e^{2\sigma} \{ P + (n-2)\lambda B \} . \end{aligned}$$

Analogously, (1.2) yields

$$f^* \bar{P} = P + (n-2)Q - \frac{n-2}{2n(n-1)} (\varepsilon \bar{\tau} e^{2\sigma} - \tau) g .$$

From the last two equations we obtain

$$Q = \frac{\varepsilon e^{2\sigma} - 1}{n-2} P + \varepsilon \lambda e^{2\sigma} B + \frac{\varepsilon \bar{\tau} e^{2\sigma} - \tau}{2n(n-1)} g .$$

Hence, using (2.6) and (2.9), we find

$$(2.11) \quad Q = \nu d\sigma \otimes d\sigma + \frac{\varepsilon \bar{\tau} e^{2\sigma} - \tau}{2n(n-1)} g ,$$

where

$$\nu = \mu \left(\frac{\varepsilon e^{2\sigma} - 1}{n} \operatorname{tr} h + \varepsilon \lambda e^{2\sigma} \right) .$$

Since $\nabla \sigma$ is isotropic, (1.3) yields $Q(X, \nabla \sigma) = 0$. Thus, applying (2.11) we conclude that

$$(2.12) \quad \varepsilon \bar{\tau} e^{2\sigma} - \tau = 0 .$$

Then, (2.11) reduces to

$$(2.11') \quad Q = \nu d\sigma \otimes d\sigma$$

or, according to (1.3) -

$$g(\nabla_X \nabla \sigma, Y) = (1 - \nu) X \sigma Y \sigma .$$

Hence, using the equation of Codazzi for g and (2.6), we derive

$$(X \mu Y \sigma - Y \mu X \sigma) Z \sigma + \frac{1}{n} \{ X \operatorname{tr} h g(Y, Z) - Y \operatorname{tr} h g(X, Z) \} = 0 .$$

Here we assume that Z is orthogonal to $\nabla \sigma$ and X , and Y is not orthogonal to Z . The result is $X \operatorname{tr} h = 0$. i.e. $\operatorname{tr} h$ is a constant. Thus, by (2.8) and (2.10), τ and $\bar{\tau}$ are also

constants. If $\bar{\tau} \neq 0$, (2.12) implies $d\sigma = 0$, which is a contradiction. Let $\bar{\tau} = 0$. According to (2.12), (2.8) and (2.10), $\tau = \text{tr } h = \lambda = 0$. By Lemma 3 c)

$$(2.13) \quad \bar{R} = e^{4\sigma} R$$

On the other hand, from $\text{tr } h = \lambda = 0$ and (1.2), (2.11'), we obtain

$$(2.14) \quad \bar{R} = \varepsilon e^{2\sigma} R$$

From (2.13) and (2.14) we find $(e^{2\sigma} - \varepsilon) R = 0$ in U_1 and hence this holds on U . Since σ can not vanish in an open subset of U , it follows $R = 0$ in U , which proves our assertion.

3. PROOF OF THEOREM 1

First we assume that there exists a point p of M such that $\|\nabla\sigma\|^2 \neq 0$ in p . Then $\|\nabla\sigma\|^2 \neq 0$ in a neighbourhood V of p . In (2.1) we assume that $X = Z = \nabla\sigma$ and that Y is orthogonal to $\nabla\sigma$. Using (2.2), we obtain $Y\lambda = 0$ in V . Hence $\nabla\lambda = \rho\nabla\sigma$ on V , where ρ is a smooth function. Using again (2.1) with $X = \nabla\sigma$ and applying (2.2), we find

$$B = \rho \left\{ \frac{1}{\|\nabla\sigma\|^2} d\sigma \otimes d\sigma - \frac{1}{n} g \right\}$$

in V . However, this contradicts the assumption that the set of nonquasi-umbilic points is dense.

So $\|\nabla\sigma\|^2 = 0$. Now, let us assume that $\nabla\sigma$ does not vanish at a point p and hence, in an open set U . By Lemma 4 a) each point of U is quasi-umbilic, which is impossible.

Consequently $\nabla\sigma$ vanishes indentially in M , i.e. σ is a constant. Then λ is also a constant. Indeed, assuming in (2.1) that V is orthogonal to X and that $Y = Z$, $g(Y, Y) \neq 0$, we obtain $X\lambda = 0$.

Since σ is a constant, (1.2) implies

$$(3.1) \quad f^*\bar{R} = \varepsilon e^{2\sigma} R .$$

Let us assume that f is not an isometry, i.e. $(\sigma, \varepsilon) \neq (0, 1)$. Then (3.1) and Lemma 3 c) yield

$$(3.2) \quad R = \alpha\varphi(h) + \beta\pi_1 ,$$

where

$$\alpha = \frac{\lambda}{\varepsilon e^{-2\sigma} - 1} , \quad \beta = \frac{\lambda^2}{\varepsilon e^{-2\sigma} - 1}$$

are constants. From (3.2), by a standard way (see e.g. [5] or [6], Example 4), we conclude that the Weyl conformal curvature tensor of M vanishes identically. So, if $n > 3$ then M is conformally flat, which is a contradiction. Let $n = 3$. Using (3.2) we find

$$(3.3) \quad S - \frac{\tau}{4} g = \alpha h + \frac{\beta}{2} g .$$

Since α, β are constants, the equation of Codazzi and (3.3) imply (1.1). Thus M is conformally flat, which is not the case. Consequently f is an isometry, i.e. $\sigma = 0, \varepsilon = 1$. Then by (3.1) and Lemma 3 c) we obtain

$$\lambda\varphi(h) + \lambda^2\pi_1 = 0 ,$$

which implies

$$(3.4) \quad \lambda\{(n-2)h + g \operatorname{tr} h\} + (n-1)\lambda^2 g = 0 .$$

But M can not be totally umbilic. So (3.4) yields $\lambda = 0$. Hence $f_*\bar{h} = h$. Since f is an isometry, this proves the theorem.

4. PROOF OF THEOREM 2

By Lemma 3 a), b) and (1.4) we conclude that $\lambda = 0$. Putting $X = \nabla\sigma$ in (2.1) and using (2.2), we obtain

$$\|\nabla\sigma\|^2 B(Y, Z) = 0 .$$

Since the nonumbilic points are dense, this implies $\|\nabla\sigma\|^2 = 0$. According to Lemma 4 b), $R = 0$ in the open set U , in which $\nabla\sigma \neq 0$. Thus the point p , in which $R \neq 0$, can not lie in the closure \bar{U} of U . Consequently, the open set $M \setminus \bar{U}$ is nonempty. Note that $d\sigma = 0$ in $M \setminus \bar{U}$. Let V be the connected component of p in $M \setminus \bar{U}$. Since σ is a constant in V , (1.2) reduces to

$$(4.1) \quad f^*\bar{R} = \varepsilon e^{2\sigma} R$$

in V . On the other hand, applying Lemma 3 c) with $\lambda = 0$, we obtain

$$(4.2) \quad f^*\bar{R} = e^{4\sigma} R .$$

From (4.1) and (4.2) we find $(e^{4\sigma} - \varepsilon)R = 0$. Since p lies in V , this implies $\sigma = 0$ (in V) and $\varepsilon = 1$. So we have $f^*\bar{g} = g, f^*\bar{h} = h$ in V . Consequently, f is a congruence of V onto $f(V)$, which completes the proof.

Remark. If the manifolds in Theorem 2 are analytic or the set of points, in which R is not zero, is dense, then f is a congruence of M onto \bar{M} .

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